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Efficient, Accurate Reanalysis for Structural Optimization

Uri Kirsch*
Technion—Israel Institute of Technology, 32000 Haifa, Israel

The combined approximations method, developed recently, is an efficient reanalysis method providing high-quality results. Through the use of this approach, the computed terms of a series expansion are used as basis vectors in a reduced basis expression. By solving a reduced system of equations, first- and second-order approximations were demonstrated in previous studies for small structures. The efficiency and the accuracy of the method are improved, and results are illustrated for larger structures. By the utilization of a Gram–Schmidt orthogonalization procedure, a new set of basis vectors is generated and normalized such that the reduced system of equations becomes uncoupled. The advantage in using the latter vectors is that all expressions for evaluating the displacements are explicit functions of the design variables. Consequently, additional vectors can be considered without modifying the calculations that have already been carried out. In addition, the uncoupled system is more well conditioned. Some considerations related to the efficiency of the solution process and the accuracy of the results are discussed, and the effect of various parameters on the accuracy is studied. Numerical results are demonstrated for several medium- and large-scale structures. It is shown that accurate and efficient approximations are achieved for very large changes in the design.

Introduction

IN structural optimization problems the behavior constraints are evaluated for successive modifications in the design. For each trial design the implicit analysis equations must be solved, and the multiple repeated analyses usually involve extensive computational effort. This difficulty motivated extensive studies on explicit approximations of the structural behavior in terms of the design variables.[1–3] The various methods may be divided as follows.

First are global multipoint approximations, such as polynomial fitting, response surface, or reduced basis methods.[4–6] The approximations are obtained by analyzing the structure at a number of design points, and they are valid for the whole design space (or, at least, large regions of it). However, global approximations may require much computational effort; therefore, they are not suitable for problems with a large number of design variables.

Second are local single-point approximations, such as the first-order Taylor series expansion or the binomial series expansion about a given design. Local approximations are based on information calculated at a single design point. These methods are most efficient, but they are effective only in cases of small changes in the design variables. For large changes in the design, the accuracy of the approximations often deteriorates, and they may become meaningless. That is, the approximations are valid only in the vicinity of a point in the design space. To improve the quality of the results, second-order approximations can be used. Another possibility is to assume the reciprocal cross-sectional areas as design variables.[7,8] A hybrid form of the direct and reciprocal approximations, which is more conservative than either[9] and has the advantage of being convex,[10] can be introduced. More accurate convex approximations can be achieved by the method of moving asymptotes.[11] One problem in using intermediate variables, such as the reciprocal variables, is that it might be difficult to select appropriate variables in cases of general optimization where geometrical, topological, or shape design variables are considered.

In general, the quality of the results and the efficiency of the calculations are conflicting factors. That is, better approximations are often achieved at the expense of more computational effort. In this paper the combined approximations (CA) method, which attempts to give global qualities to local approximations, will be considered. One approach to introduce combined approximations is to scale the initial design and the low-order terms such that the results are improved. It has been shown[12] that scaling procedures are useful for various types of design variables and behavior functions. In particular, simplified approximations can be achieved for homogeneous functions.[13] The concept of scaling has been extended to include high-order terms of the CA method,[14,15] thereby allowing improved results. The effectiveness of the CA method in problems of cross-section optimization as well as geometrical and topological optimization, has been demonstrated elsewhere.[14–19] By the use of this approach, the computed terms of a series expansion are used as basis vectors in a reduced basis expression to obtain an effective solution procedure. It has been shown previously that the unknown coefficients can readily be determined by solving a reduced set of the analysis equations. The method is based on results of a single exact analysis, and it is suitable for different types of structure and design variables. Calculation of derivatives is not required, and the procedure can readily be used with a general finite element program.

Initially, the CA method has been used only for linear analysis models. Recently, it has been found that the method also might prove useful for more complex structural response. Specifically, significant reductions in the computational effort involved in solution of nonlinear analysis problems were reported.[20] In addition, the method has been used successfully in the solution of eigenvalue problems in damage analysis of frames.[21] In both studies accurate results and significant savings in the computational effort were reported. A detailed discussion on applications of the CA method in a large variety of problems, including nonlinear and dynamic analysis, is given elsewhere.[19]

In this paper, the efficiency and the accuracy of the CA method are improved. By utilizing a Gram–Schmidt orthogonalization procedure, a new set of basis vectors is generated and normalized such that the reduced system becomes uncoupled. For any assumed number of basis vectors, the results obtained by considering either the original set of basis vectors and the reduced set of equations or the new set of uncoupled basis vectors are identical. The advantage in using the latter vectors is that all expressions for evaluating the displacements are explicit functions of the design variables. Consequently, additional vectors can be considered without modifying the calculations that already were carried out. In addition, the uncoupled system is more well conditioned.

Some considerations related to the efficiency of the solution process and the accuracy of the results are discussed. The effect of the number of terms in the series, the magnitude of changes in the design, and the direction of move in the design space on the accuracy are studied. Numerical results are demonstrated for several medium- and large-scale structures.

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*Professor, Department of Civil Engineering; currently Visiting Professor, Department of Mechanical Engineering and Applied Mechanics, University of Michigan, 2250 G.G. Brown Building, Ann Arbor, MI 48109.

Efficient, Accurate Approximations

In this section the CA method is developed further to obtain efficient, accurate approximations. The reanalysis problem under consideration is first formulated. The terms of the series that are used as basis vectors are then developed, and the reduced basis expressions are introduced. Finally, it is shown how a Gram–Schmidt orthogonalization procedure is utilized to generate the new set of basis vectors such that the reduced system becomes uncoupled.

Formulation of Reanalysis

The problem considered in this study may be stated as follows:

1) Given an initial design and the corresponding stiffness matrix $\boldsymbol{K}^*$, the displacements $\boldsymbol{r}^*$ are computed by the equilibrium equations

$$\boldsymbol{K}^*\boldsymbol{r}^* = \boldsymbol{R} \tag{1}$$

The load vector $\boldsymbol{R}$ is usually assumed to be independent of the design variables. In addition, the stiffness matrix $\boldsymbol{K}^*$ is often given from the initial analysis in the decomposed form

$$\boldsymbol{K}^* = \boldsymbol{U}^{*T}\boldsymbol{U}^* \tag{2}$$

where $\boldsymbol{U}^*$ is an upper triangular matrix.

2) Assume a change in the design so that the modified stiffness matrix is

$$\boldsymbol{K} = \boldsymbol{K}^* + \Delta\boldsymbol{K} \tag{3}$$

where $\Delta\boldsymbol{K}$ is the change in the stiffness matrix due to the change in the design.

3) The object is to find efficient and accurate approximations of the modified displacements $\boldsymbol{r}$ due to various changes in the design, without solving the modified analysis equations

$$\boldsymbol{K}\boldsymbol{r} = (\boldsymbol{K}^* + \Delta\boldsymbol{K})\boldsymbol{r} = \boldsymbol{R} \tag{4}$$

Once the displacements are evaluated, the stresses can readily be determined by the explicit stress-displacement relations. Thus, the presented approximations of $\boldsymbol{r}$ are intended only to replace the set of implicit analysis equations (4).

Series Approximations

Although Taylor series is the most commonly used approximation in structural optimization, the binomial series is considered in this study. The advantage of using the latter series is that, high-order terms can readily be calculated. In addition, unlike the Taylor series, calculation of derivatives is not required. This makes the method more attractive in general applications, where derivatives are not easy to calculate. It has been shown[3] that for homogeneous displacement functions the Taylor series and the binomial series are equivalent.

The binomial series approximations can be obtained by rearranging Eq. (4) to read

$$\boldsymbol{K}^*\boldsymbol{r} = \boldsymbol{R} - \Delta\boldsymbol{K}\boldsymbol{r} \tag{5}$$

By writing this equation as the recurrence relation

$$\boldsymbol{K}^*\boldsymbol{r}^{(k+1)} = \boldsymbol{R} - \Delta\boldsymbol{K}\boldsymbol{r}^{(k)} \tag{6}$$

where $\boldsymbol{r}^{(k+1)}$ is the value of $\boldsymbol{r}$ after the kth cycle, and by assuming the initial value $\boldsymbol{r}^{(1)} = \boldsymbol{r}^*$, the following series approximations are obtained:

$$\boldsymbol{r} = \boldsymbol{r}^* - \boldsymbol{B}\boldsymbol{r}^* + \boldsymbol{B}^2\boldsymbol{r}^* - \cdots + \boldsymbol{B}^{s-1}\boldsymbol{r}^* \tag{7}$$

In this equation, s is the number of terms in the series, and matrix $\boldsymbol{B}$ is defined by

$$\boldsymbol{B} \equiv \boldsymbol{K}^{*-1}\Delta\boldsymbol{K} \tag{8}$$

Denoting

$$\boldsymbol{r}_1 = \boldsymbol{r}^*, \qquad \boldsymbol{r}_2 = -\boldsymbol{B}\boldsymbol{r}^*, \qquad \boldsymbol{r}_3 = \boldsymbol{B}^2\boldsymbol{r}^*, \ldots, \boldsymbol{r}_s = -\boldsymbol{B}^{s-1}\boldsymbol{r}^* \tag{9}$$

and substituting Eqs. (9) into Eq. (7) yields

$$\boldsymbol{r} = \boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3 + \cdots + \boldsymbol{r}_s \tag{10}$$

In general, local series approximations are not suitable for large changes in the design because the quality of the results is insufficient. Problems of slow convergence and even divergence of the series might be encountered, as will be shown later.

Reduced Basis Approximations

The reduced basis method[3–5] usually involves analysis of the structure at a number of design points; therefore, it may be classified as a global approximation. A basic question in using the method lies in the choice of an appropriate set of the linearly independent vectors that span the design variables space. Displacement vectors of previously analyzed designs can be used, but it should be emphasized that an intuitive choice may not lead to satisfactory approximations. In addition, calculation of the basis vectors requires several exact analyses of the structure for the basis design points, which involve extensive computational effort.

In the CA approach[14–19] it is assumed that the displacement vector of a new design can be approximated by a linear combination of the s linearly independent basis vectors of Eq. (9) (where s is assumed to be much smaller than the number of degrees of freedom m). The advantage is that the efficiency of local series approximations and the improved quality of global reduced basis approximations are combined to achieve an effective solution procedure. By the use of this approach, the displacements are approximated as

$$\boldsymbol{r} = y_1\boldsymbol{r}_1 + y_2\boldsymbol{r}_2 + \cdots + y_s\boldsymbol{r}_s = \boldsymbol{r}_B\boldsymbol{y} \tag{11}$$

where

$$\boldsymbol{r}_B = \boldsymbol{r}_1, \ldots, \boldsymbol{r}_s, \qquad \boldsymbol{y} = \begin{Bmatrix} y_1 \\ \vdots \\ y_s \end{Bmatrix} \tag{12}$$

and $\boldsymbol{y}$ is a vector of coefficients to be determined. Substituting Eq. (11) into the modified analysis equations (4) and premultiplying by $\boldsymbol{r}_B^T$ yields

$$\boldsymbol{r}_B^T\boldsymbol{K}\boldsymbol{r}_B\boldsymbol{y} = \boldsymbol{r}_B^T\boldsymbol{R} \tag{13}$$

Introducing the notation

$$\boldsymbol{K}_R = \boldsymbol{r}_B^T\boldsymbol{K}\boldsymbol{r}_B, \qquad \boldsymbol{R}_R = \boldsymbol{r}_B^T\boldsymbol{R} \tag{14}$$

and substituting into Eq. (13) gives

$$\boldsymbol{K}_R\boldsymbol{y} = \boldsymbol{R}_R \tag{15}$$

For cases where s is much smaller than the number of degrees of freedom m, the approximate displacement vector can be evaluated by solving the smaller $(s \times s)$ system in Eq. (15) for $\boldsymbol{y}$ instead of computing the exact solution by solving the large $(m \times m)$ system in Eq. (4). The final displacements are then computed for the given $\boldsymbol{y}$ by Eq. (11).

Uncoupled Basis Vectors

Leu and Huang[20] proposed recently to use the CA method for nonlinear analysis of structures. By the utilization of a Gram–Schmidt orthogonalization procedure to generate a new set of basis vectors, the reduced system of analysis equations becomes uncoupled; thus, possible ill conditioning is avoided in nonlinear analysis.

In this paper, a Gram–Schmidt orthogonalization procedure will be used to obtain explicit, accurate approximations for the reanalysis problem under consideration. From Eq. (14), it can be observed that the elements K_{Rij} of the reduced stiffness matrix $\boldsymbol{K}_R$ are given by

$$K_{Rij} = \boldsymbol{r}_i^T\boldsymbol{K}\boldsymbol{r}_j \tag{16}$$

The object now is to transform the reduced system of Eqs. (15) into uncoupled set of equations. This can be done by generating a set of new basis vectors $\boldsymbol{V}_i$ $(i = 1, \ldots, s)$, substituting the original ones $\boldsymbol{r}_i$, such that for any two vectors $\boldsymbol{V}_i$ and $\boldsymbol{V}_j$

$$\boldsymbol{V}_i^T\boldsymbol{K}\boldsymbol{V}_j = \delta_{ij} \tag{17}$$

where δ_{ij} is the Kronecker delta. The new basis vectors, which are linear combinations of the original vectors, are generated as follows. We start by choosing the first normalized vector $\boldsymbol{V}_1$

$$\boldsymbol{V}_1 = \left|\boldsymbol{r}_1^T \boldsymbol{K} \boldsymbol{r}_1\right|^{-\frac{1}{2}} \boldsymbol{r}_1 \tag{18}$$

To generate the second normalized vector $\boldsymbol{V}_2$, we first define the nonnormalized vector $\bar{\boldsymbol{V}}_2$, which is a linear combination of $\boldsymbol{V}_1$ and $\boldsymbol{r}_2$, by

$$\bar{\boldsymbol{V}}_2 = \boldsymbol{r}_2 - \alpha \boldsymbol{V}_1 \tag{19}$$

where α is chosen such that the orthogonality condition $\bar{\boldsymbol{V}}_2^T \boldsymbol{K} \boldsymbol{V}_1 = 0$ [Eq. (17)] is satisfied. Substituting Eq. (19) into the latter condition yields

$$\bar{\boldsymbol{V}}_2^T \boldsymbol{K} \boldsymbol{V}_1 = \boldsymbol{r}_2^T \boldsymbol{K} \boldsymbol{V}_1 - \alpha \boldsymbol{V}_1^T \boldsymbol{K} \boldsymbol{V}_1 = 0 \tag{20}$$

Because $\boldsymbol{V}_1^T \boldsymbol{K} \boldsymbol{V}_1 = 1$ [Eq. (18)], Eq. (20) becomes $\alpha = \boldsymbol{r}_2^T \boldsymbol{K} \boldsymbol{V}_1$. Substitution into Eq. (19) gives

$$\bar{\boldsymbol{V}}_2 = \boldsymbol{r}_2 - \left(\boldsymbol{r}_2^T \boldsymbol{K} \boldsymbol{V}_1\right) \boldsymbol{V}_1 \tag{21}$$

Finally, normalizing $\bar{\boldsymbol{V}}_2$ yields

$$\boldsymbol{V}_2 = \left|\bar{\boldsymbol{V}}_2^T \boldsymbol{K} \bar{\boldsymbol{V}}_2\right|^{-\frac{1}{2}} \bar{\boldsymbol{V}}_2 \tag{22}$$

Additional new basis vectors are generated in a similar way. The resulting general expressions for $i = 2, \ldots, s$ are

$$\bar{\boldsymbol{V}}_i = \boldsymbol{r}_i - \sum_{j=1}^{i-1} \left(\boldsymbol{r}_i^T \boldsymbol{K} \boldsymbol{V}_j\right) \boldsymbol{V}_j, \qquad \boldsymbol{V}_i = \left|\bar{\boldsymbol{V}}_i^T \boldsymbol{K} \bar{\boldsymbol{V}}_i\right|^{-\frac{1}{2}} \bar{\boldsymbol{V}}_i \tag{23}$$

where $\bar{\boldsymbol{V}}_i$ and $\boldsymbol{V}_i$ are the ith nonnormalized and normalized vectors, respectively.

It will be shown now how the new basis vectors $\boldsymbol{V}_i$ are used to evaluate the displacements. Assuming the latter vectors, it can be observed from Eq. (17) that the new reduced matrix is a unit matrix $\boldsymbol{I}$. By defining the matrix $\boldsymbol{V}_B$ and the vector of new coefficients z by

$$\boldsymbol{V}_B = \boldsymbol{V}_1, \ldots, \boldsymbol{V}_s, \qquad z = \begin{Bmatrix} z_1 \\ \vdots \\ z_s \end{Bmatrix} \tag{24}$$

then the reduced system of Eq. (15) becomes

$$\boldsymbol{I} z = z = \boldsymbol{V}_B^T \boldsymbol{R} \tag{25}$$

As expected, this system is uncoupled, and the coefficients z can be determined directly. Thus, the final displacements are given by the explicit expression

$$\boldsymbol{r} = \boldsymbol{V}_B z = \boldsymbol{V}_B \left(\boldsymbol{V}_B^T \boldsymbol{R}\right) \tag{26}$$

In summary, the proposed solution procedure consists of the following steps:

1) The modified stiffness matrix $\boldsymbol{K}$ is introduced.
2) The basis vectors $\boldsymbol{r}_i$ are calculated by Eq. (9).
3) The new set of basis vectors $\boldsymbol{V}_i (i = 1, \ldots, s)$ is generated by Eqs. (18) and (23).
4) The final displacements are evaluated by Eq. (26).

Note that, for any assumed number of basis vectors, the results obtained by considering either the original set of basis vectors and the reduced set of equations, or the new set of uncoupled basis vectors, are identical. The advantage in using the latter vectors is that all expressions for evaluating the displacements are explicit functions of the design variables. Consequently, additional vectors can be considered without modifying the calculations that already were carried out. In addition, the uncoupled system is more well conditioned.

Computational Considerations

Computational Effort

The computational effort involved in the CA is somewhat greater compared with conventional series approximations, such as the Taylor series or the binomial series. The series approximations involve only calculation of the original basis vectors. The CA require, in addition, calculation of the modified stiffness matrix $\boldsymbol{K}$, generation of the new basis vectors $\boldsymbol{V}_i$, and some extra algebraic operations. These operations increase slightly the computational cost, but the resulting expressions are still explicit. In addition, accurate approximations can be achieved in cases where series approximations provide poor or meaningless results.

It has been shown[3] that calculation of the basis vectors involves only forward and backward substitutions if $\boldsymbol{K}^*$ is given in the decomposed form of Eq. (2). The calculation of $\boldsymbol{r}_2$, for example, is carried out by means of this equation [see Eq. (9)]:

$$\boldsymbol{K}^* \boldsymbol{r}_2 = -\Delta \boldsymbol{K} \boldsymbol{r}^* \tag{27}$$

We first solve for $\boldsymbol{t}$ by the forward substitution

$$\boldsymbol{U}^{*T} \boldsymbol{t} = -\Delta \boldsymbol{K} \boldsymbol{r}^* \tag{28}$$

Then, $\boldsymbol{r}_2$ is calculated by the backward substitution

$$\boldsymbol{U}^* \boldsymbol{r}_2 = \boldsymbol{t} \tag{29}$$

Similarly, the ith basis vector $\boldsymbol{r}_i$ is calculated from

$$\boldsymbol{K}^* \boldsymbol{r}_i = -\Delta \boldsymbol{K} \boldsymbol{r}_{i-1}, \qquad i = 2, \ldots, s \tag{30}$$

In summary, calculation of each term of the binomial series involves only forward and backward substitutions if $\boldsymbol{K}^*$ is given in the decomposed form of Eq. (2).

As noted earlier, the new basis vectors are explicit functions of the original basis vectors [Eqs. (18) and (23)]. Therefore, once the original vectors are determined, calculation of the new vectors and the final displacements [Eq. (26)] is straightforward.

The efficiency of reanalysis by the CA method, compared with complete analysis of the modified design, can be measured by various criteria, for example the CPU effort or the number of algebraic operations. It is then possible to relate the computational effort to various parameters such as the number of degrees of freedom, the number of basis vectors considered, and the accuracy of the results. In a recent study it has been found[20] that, using the CA method for a complete accurate nonlinear analysis of a space frame with about 300 degrees of freedom, the resulting CPU effort has been reduced by more than 60%. Certainly, to quantify precisely the computational savings, further programming efforts and comparative studies are needed.

Convergence of the Series

Problems of slow convergence or divergence may be encountered in applying the series approximations of Eq. (7). The series converges if and only if

$$\lim_{k \to \infty} \boldsymbol{B}^k = \boldsymbol{0} \tag{31}$$

A sufficient criterion for the convergence of the series is that

$$\|\boldsymbol{B}\| \leq 1 \tag{32}$$

where $\|\boldsymbol{B}\|$ is the norm of $\boldsymbol{B}$. It can be shown that

$$\rho(\boldsymbol{B}) \leq \|\boldsymbol{B}\| \tag{33}$$

in which $\rho(\boldsymbol{B})$ is the spectral radius of matrix $\boldsymbol{B}$, defined as the largest eigenvalue. From Eqs. (32) and (33), a sufficient condition for convergence is

$$\rho(\boldsymbol{B}) \leq 1 \tag{34}$$

The number of basis vectors considered in evaluating the displacements has a significant influence on the accuracy and the efficiency of the calculations. In the CA method, the series terms are gradually decreased; thus consideration of more basis vectors will improve the accuracy of the results.

Three typical cases related to convergence of the series approximations (SA) of Eq. (7) will be demonstrated later in this paper:

1) Fast convergence of the SA happens only in cases where the change in the design is relatively small and higher-order terms rapidly approach zero. High accuracy can be achieved by both the SA [Eq. (7)] and the CA method with only a small number of terms.

2) In the slow convergence of the SA case, the elements of matrix $\boldsymbol{B}$ and the original basis vectors are reduced gradually. To achieve sufficient accuracy by the SA, a large number of terms might be necessary. Using the CA method, on the other hand, a small number of terms might be sufficient.

3) In the divergence of the SA case, the elements of matrix $\boldsymbol{B}$ and the original basis vectors are increased gradually. The SA provide meaningless results and cannot be used, whereas accurate results are achieved by the CA method.

Errors Evaluation

To evaluate the quality of the results for any assumed number of basis vectors, we can substitute the approximate displacements [Eq. (26)] into the modified analysis equations (4). The errors in the equilibrium equations, given by

$$\boldsymbol{E} = \boldsymbol{K}\boldsymbol{r} - \boldsymbol{R} \tag{35}$$

indicate the discrepancy in satisfying the modified equilibrium conditions due to the approximate displacements. Thus, $\boldsymbol{E}$ can be used to evaluate the quality of the approximations. A possible criterion for acceptable approximations is

$$\boldsymbol{E}^L \leq \boldsymbol{E} \leq \boldsymbol{E}^U \tag{36}$$

where $\boldsymbol{E}^L$ and $\boldsymbol{E}^U$ are the predetermined acceptable bounds on $\boldsymbol{E}$. Note that the amount of extra calculations involved in evaluating the errors $\boldsymbol{E}$ is minor because both $\boldsymbol{K}$ and the approximate displacements are already known.

Alternatively, by assuming that the solution converges and the denoting of the approximate displacements obtained with k basis vectors as $\boldsymbol{r}^{(k)}$, then it is possible to use the following criterion for the adequacy of the results:

$$\varepsilon \boldsymbol{r}^L \leq \boldsymbol{r}^{(k)} - \boldsymbol{r}^{(k-1)} \leq \varepsilon \boldsymbol{r}^U \tag{37}$$

in which $\varepsilon \boldsymbol{r}^L$ and $\varepsilon \boldsymbol{r}^U$ are the predetermined small values. In cases where the accuracy, as defined by the preceding criteria, is not sufficient, it is possible to improve the results by assuming more basis vectors.

Parameters Affecting the Results

The effect of some parameters on the accuracy of the approximations is demonstrated subsequently. In structural optimization the modified design variables $\boldsymbol{X}$ are often defined in terms of the initial design $\boldsymbol{X}^*$, a given direction vector in the design space $\Delta\boldsymbol{X}^*$, and a step size variable a by

$$\boldsymbol{X} = \boldsymbol{X}^* + a\Delta\boldsymbol{X}^* \tag{38}$$

The angle θ between the vector of the modified design and the vector of initial design (Fig. 1) is determined by

$$\cos\theta = \frac{\boldsymbol{X}^T\boldsymbol{X}^*}{|\boldsymbol{X}||\boldsymbol{X}^*|} \tag{39}$$

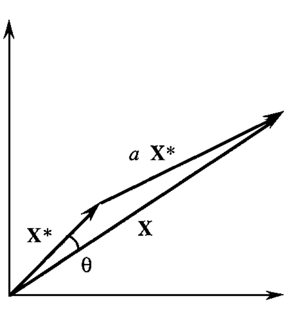

Fig. 1 Plane of vectors X and X^*.

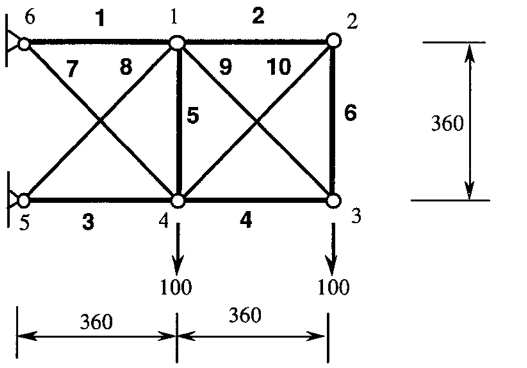

Fig. 2 Truss with 10 bars.

in which $|\boldsymbol{X}|$ denotes absolute value of $\boldsymbol{X}$. The effect of the following parameters on the accuracy of the approximations will be demonstrated subsequently by numerical examples: the number of terms in the series (s), the magnitude of changes in the design (the step size a), and the nature of changes (the direction vector $\Delta\boldsymbol{X}^*$ or the angle θ). It will be shown that a small number of terms is sufficient to achieve accurate approximations. As to the effect of a on the accuracy, it is clear that for smaller a values the approximations are more accurate. In regard to the effect of the direction of movement in the design space, previous experience has shown that accurate results are achieved by the CA method with only 2 or 3 terms, for very large changes in the design, in cases where the angle θ is relatively small.

Numerical Examples

General Observations

In all examples the design variables are the members' cross-sectional areas, and the initial cross sections equal unity. The objective of the examples presented in this section is to illustrate the effect of various parameters on the accuracy of the results achieved by the CA method.

Consider the 10-bar truss shown in Fig. 2 and subjected to a single loading condition of two concentrated loads. The modulus of elasticity is $E = 30{,}000$, and the eight analysis unknowns are the horizontal and the vertical displacements at joints 1–4, respectively. The stress constraints are $-25.0 \leq \sigma \leq 25.0$, and the minimum size constraints are $0.001 \leq \boldsymbol{X}$. By assuming the weight as an objective function, the resulting optimal design is

$$\boldsymbol{X}^T_{\text{opt}} = \{8.0, 0.001, 8.0, 4.0, 0.001, 0.001, 5.66, 5.66, 5.66, 0.001\}$$

Assume the line from the initial design to the optimal design, given by Eq. (38), where $\Delta\boldsymbol{X}^*$ is defined as

$$\Delta\boldsymbol{X}^{*T} = \{7.0, -0.999, 7.0, 3.0, -0.999, \\ -0.999, 4.66, 4.66, 4.66, -0.999\}$$

For $a = 1.0$ (the optimum) the changes are very large: members 1 and 3 are increased by 700%, member 4 is increased by 300%, members 7–9 are increased by 466%, and, at the same time the topology is practically changed by eliminating members 2, 5, 6, and 10 and joint 2 (displacements 3 and 4).

To illustrate the effect of a, three typical cases have been considered: 1) small change in the variables, $a = 0.01$; 2) medium change in the design variables, $a = 0.1$; and 3) large change in the design variables, $a = 1.0$ (the optimum). Results obtained by the SA of Eq. (7) and the CA method for various numbers of terms (basis vectors) are summarized in Table 1 and demonstrated in Figs. 3–6. It may be observed that the three a values considered correspond to the three cases of convergence of the SA discussed earlier, that is, fast convergence (for $a = 0.01$), slow convergence (for $a = 0.1$), and divergence of the series (for $a = 1$).

To illustrate the effect of the direction of move in the design space, assume the three cases of change in the design variables shown in Table 2, with different θ values [see Eq. (39) and Fig. 1]. The results obtained by two-term approximations are summarized in Table 3.

Table 1 Displacements obtained by the SA and the CA methods

	SA [Eq. (7)]				CA				
Step size	2[a]	3	4	5	2	3	4	5	Exact
$a = 0.01$	2.18	2.19			2.19				2.19
(small)	5.26	5.28			5.28				5.28
	2.65	2.66			2.66				2.66
	11.95	11.99			11.99				11.99
	−3.00	−3.01			−3.01				−3.01
	12.42	12.46			12.46				12.46
	−2.29	−2.30			−2.30				−2.30
	5.68	5.69			5.69				5.69
$a = 0.1$	0.69	1.85	1.04	1.61	1.36	1.37			1.37
(medium)	2.38	4.28	3.12	3.85	3.59	3.56			3.56
	1.07	2.26	1.44	2.01	1.76	1.77			1.77
	5.59	9.96	7.12	9.01	8.23	8.25			8.25
	−1.40	−2.58	−1.76	−2.33	−2.06	−2.10			−2.10
	5.97	10.36	7.52	9.42	8.62	8.65			8.65
	−0.75	−1.94	−1.11	−1.69	−1.44	−1.45			−1.45
	2.69	4.61	3.44	4.18	3.92	3.89			3.89
$a = 1.0$	Divergence of the SA [Eq. (7)]				0.28	0.29	0.29	0.30	0.30
(large)					0.90	0.84	0.88	0.90	0.90
					0.41	0.45	0.47	0.49	0.49
					2.10	2.17	2.19	2.21	2.21
					−0.53	−0.61	−0.62	−0.60	−0.60
					2.24	2.34	2.37	2.40	2.40
					−0.30	−0.31	−0.31	−0.30	−0.30
					1.01	0.95	0.93	0.90	0.90

[a]Number of terms.

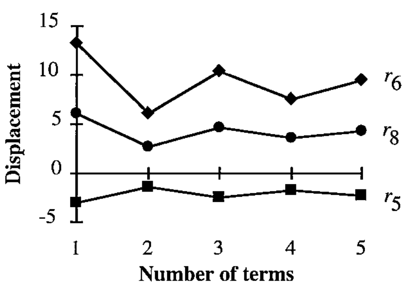

Fig. 3a SA at $a = 0.1$.

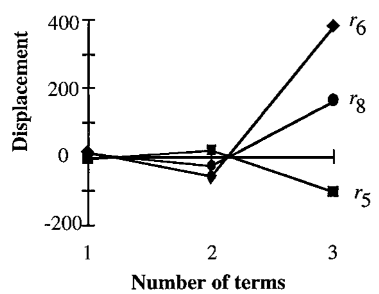

Fig. 4a SA at $a = 1.0$.

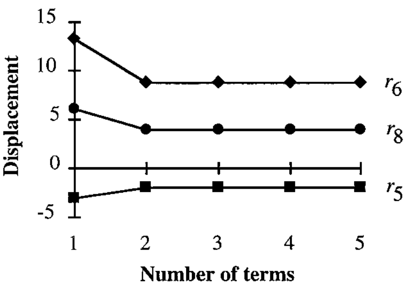

Fig. 3b CA at $a = 0.1$.

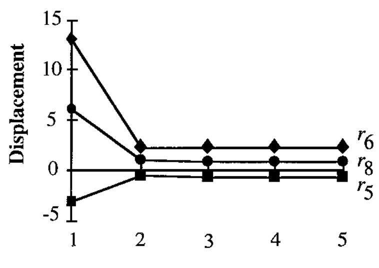

Fig. 4b CA at $a = 1.0$.

The following general observations, supported by the numerical results, have been made:

1) The effect of specific parameters on various displacements is similar. Tables 1 and 3 illustrate the effect of the method of solution (the SA and the CA), the number of terms considered (s), the magnitude of change (the step size a), and the direction of move in the design space (the angle θ). Because results for the various displacements indicate similar effects, in Figs. 5 and 6 only results for the vertical displacement at joint 3 (r_6) are demonstrated.

2) The CA method provides much better results than the SA. Furthermore, accurate results are obtained even in cases where the series diverges. This result has been demonstrated for various cases of s, a, and θ values.

3) When more terms in the CA method are considered, the solution becomes more accurate. In the examples shown, a small number of terms is sufficient to achieve an accurate solution. This observation does not hold for the SA in cases where the series diverges.

4) For a given direction $\Delta \boldsymbol{X}^*$, more terms are required to achieve a certain accuracy the larger the step size is. Again, this observation does not hold for the SA, in cases where the series diverges. Table 1 and Figs. 3–6 show fast convergence for the CA method in all cases. For the SA, slow convergence is obtained for $a = 0.1$, and divergence occurs for $a = 1.0$. In this example $\rho(\boldsymbol{B}) = 7a$, that is, the sufficient condition for convergence [Eq. (34)] is satisfied only for relatively small changes in the design (corresponding to $a \leq \frac{1}{7}$).

5) Better accuracy is achieved for directions corresponding to small θ values. The results in Table 3 for two-term approximations show that in case A, where a uniform large change of $\Delta X = 9$ is assumed for all cross sections, the series approximations provide

Table 2 Data: various direction vectors

Case[a]	θ	$\Delta X_1 = \cdots = \Delta X_4$	$\Delta X_5 = \cdots = \Delta X_{10}$
A	0	9.0	9.0
B	6.3	9.0	7.0
C	47.0	0.9	−0.9

[a]Case A, large ΔX, $\theta = 0$; case B, large ΔX, small θ; case C, small ΔX, large θ.

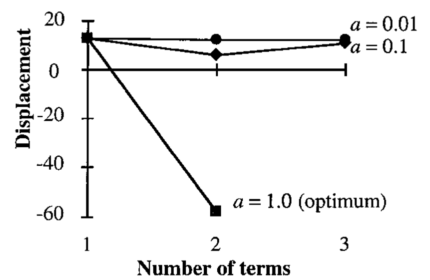

Fig. 5a SA: r_6 (vertical displacement at joint 3).

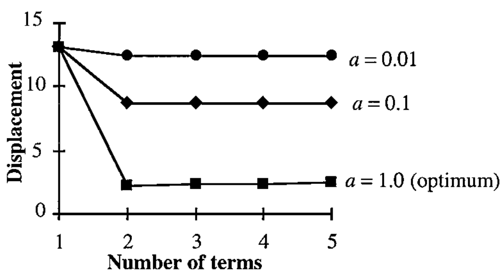

Fig. 5b CA: r_6 (vertical displacement at joint 3).

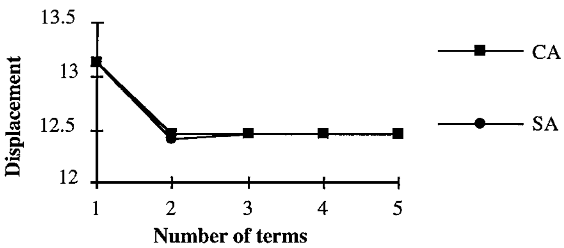

Fig. 6a Vertical displacement at joint 3 (r_6) at $a = 0.01$.

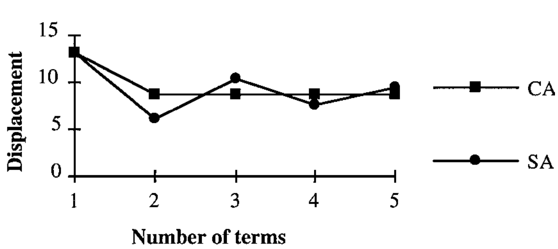

Fig. 6b Vertical displacement at joint 3 (r_6) at $a = 0.1$.

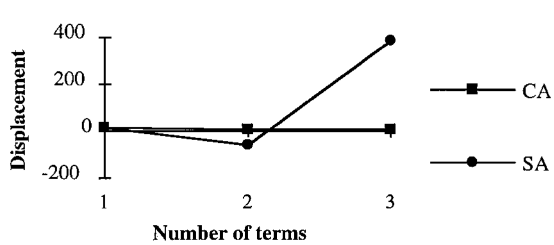

Fig. 6c Vertical displacement at joint 3 (r_6) at $a = 1.0$ (optimum).

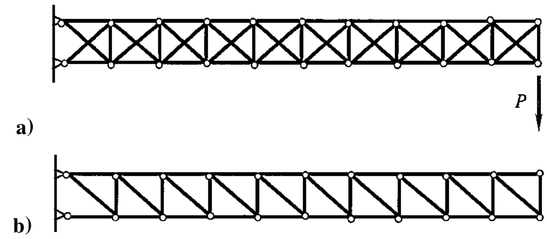

Fig. 7 Truss with 50 bars.

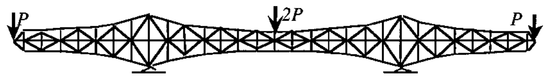

Fig. 8 Truss with 204 bars.

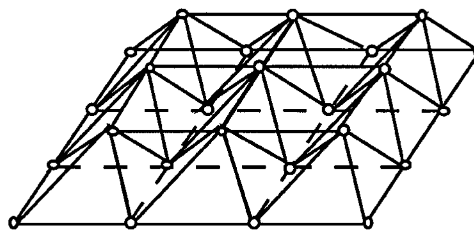

Fig. 9 Segment of double-layer space truss.

meaningless results. Because the modified design is a scaled design ($X = 10$, $\theta = 0$), the exact solution is achieved by the CA method. Similar observations have been made in case B (a small θ value) where good results are achieved by the CA. In case C (a large θ value), the CA still provide relatively good results.

Medium- and Large-Scale Structures

In this section, results achieved by the CA method for some structures with a larger number of degrees of freedom will be demonstrated. In the examples presented, the errors are related to the exact solution, which was calculated for purposes of comparison. In general, the exact solution is not known a priori, but the errors involved in the approximations can be evaluated by the expressions of Eqs. (35–37).

Consider first the 50-bar cantilever truss shown in Fig. 7a, subjected to a single load at the end. By eliminating 10 diagonal members to obtain the topology shown in Fig. 7b and by assuming only two basis vectors (two-term approximations), a near exact solution has been achieved. The angle θ in this case is 26.3 deg. Near exact solutions with only two basis vectors have been achieved also for numerous other cases with random changes in cross sections for small θ values ($\theta \leq 30$ deg). By assuming random changes in cross sections with no limitations on θ, the following maximum errors have been achieved[22]: maximum error of 1% with 5 basis vectors and maximum error of 0.1% with 6 basis vectors.

Consider the 204-bar truss (shown in Fig. 8) subjected to three concentrated loads. By assuming 100 cases of random changes in all cross sections, the following maximum errors have been achieved[22]: maximum error of 1% with six or seven basis vectors and maximum error of 0.1% with eight or nine basis vectors.

Consider the typical double layer segment shown in Fig. 9, consisting of two horizontal layers connected by diagonals. Various rectangular large-scale trusses, consisting of the typical segments, supported along the four edges and subjected to uniformly distributed loads, have been solved. Assuming 100 cases of random changes in all cross sections, the following maximum errors have been achieved[22]: For the 356-bar truss, there was maximum error of 1% with 8 or 9 basis vectors and maximum error of 0.1% with 9 or 10 basis vectors. For the 968-bar truss, there was maximum error of 1% with 10 or 11 basis vectors and maximum error of 0.1% with 11 or 12 basis vectors.

The results indicate that small errors have been achieved for various structures with a relatively small number of basis vectors. Moreover, the number of vectors needed to achieve a certain accuracy

Table 3 Results, two-term approximations, various direction vectors

Case	Method	Displacements 1	2	3	4	5	6	7	8
A	SA				Meaningless results				
	CA	0.23	0.56	0.28	1.26	−0.31	1.31	−0.25	0.60
	Exact	0.23	0.56	0.28	1.26	−0.31	1.31	−0.25	0.60
B	SA				Meaningless results				
	CA	0.23	0.64	0.28	1.39	−0.32	1.45	−0.25	0.69
	Exact	0.23	0.64	0.28	1.39	−0.32	1.45	−0.25	0.69
C	SA	0.21	6.31	0.22	12.02	−0.37	10.89	−0.27	4.95
	CA	1.13	33.16	1.16	52.59	−1.98	57.17	−1.38	37.09
	Exact	1.23	33.27	1.47	52.75	−1.69	57.29	−1.30	37.14

is only slightly increased with the numbers of members and degrees of freedom.

Concluding Remarks

The CA method, developed recently, is an efficient reanalysis method, providing high-quality results for structural optimization. In this approach, the computed terms of a series expansion are used as basis vectors in a reduced basis expression. By solving a reduced system of equations, first-order and second-order approximations were demonstrated in previous studies for small structures. The method is based on results of a single exact analysis, and it is suitable for different types of structure and design variables. Calculation of derivatives is not required, and the method can readily be used with a general finite element program.

In this paper, the efficiency and the accuracy of the method are improved, and results are demonstrated for various structures. By utilizing a Gram–Schmidt orthogonalization procedure, a new set of basis vectors is generated and normalized such that the reduced system of equations becomes uncoupled. It is shown that, for any assumed number of basis vectors, the results obtained by considering either the original set of basis vectors and the reduced set of equations, or the new set of uncoupled basis vectors, are identical. The advantage in using the latter vectors is that all expressions for evaluating the displacements are explicit functions of the design variables. Therefore, additional basis vectors can be considered without modifying the calculations that already were carried out. In addition, the uncoupled system is more well conditioned.

The computational effort involved in the CA is somewhat greater compared with conventional series approximations, such as the Taylor series or the binomial series, but the resulting expressions are still explicit, and accurate approximations can be achieved in cases where series approximations provide poor or meaningless results.

Some considerations related to the efficiency of the solution process and the accuracy of the results are discussed. The effect of the number of terms in the series, the magnitude of changes in the design, and the direction of move in the design space on the accuracy are studied.

Numerical results have been demonstrated for several medium- and large-scale structures. It has been shown that accurate and efficient approximations are achieved for very large changes in the design variables.

The CA method also might prove useful for more complex structural response. A detailed discussion on applications of the CA method in a large variety of problems, including nonlinear and dynamic analysis, is given elsewhere.[19]

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A. D. Belegundu
Associate Editor